NL Hausdorff : Kapsel 43 : Fasz. 778

Kai-Lai Chung, Sur un théorème de M. Gumbel

Hs. Ms. – [Bonn], 3. 11. 1941. – 2 Bll.

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Kai-Lai Chung, Sur un théorème de M. Gumbel, C. R. 210, 620–621 (1940)

Es seien n Ereignisse $1, 2, \ldots, n$ betrachtet; $m \leq k \leq n$. Dann ist

$$p_m(1, 2, \ldots, n) \leq \frac{1}{\binom{n-m}{k-m}} \sum p_m(\nu_1, \nu_2, \ldots, \nu_k), \tag{0}$$

wo $p_m(\nu_1, \nu_2, \ldots, \nu_k)$ die Wahrsch. ist, dass von den Ereignissen $\nu_1, \nu_2, \ldots, \nu_k$ mindestens m eintreten, und die Summe über alle k-gliedrigen Kombinationen aus der Reihe $1, 2, \ldots, n$ erstreckt ist.

(Resultat in Math. Reviews 2, p. 106 (1941) angegeben, Beweis von mir).

$\sigma_\gamma = \{\nu_1, \ldots, \nu_\gamma\}$ $\quad (\nu_1 < \cdots < \nu_\gamma)$ sei eine γ-gliedrige Menge $\subset \{1, 2, \ldots, n\}$ und (σ_γ) die W., dass genau die Ereignisse $\nu_1, \ldots, \nu_\gamma$ und keine andern eintreten (kurz, dass das Ereignis σ_γ eintrete); diese σ_γ sind disjunkt. $p_\gamma = \sum(\sigma_\gamma)$ bei festem γ ist die W., dass genau γ von den Ereignissen $1, \ldots, n$ eintreten. Also

$$p_m(1, 2, \ldots, n) = p_m + p_{m+1} + \cdots + p_n. \tag{1}$$

Die Anzahl der Elementarwahrsch. (σ_γ), aus denen sich p_γ zusammensetzt, ist $\binom{n}{\gamma}$.

σ_α sei eine α-gliedrige Menge $\subset (1, 2, \ldots, k)$ und σ_β eine β-gliedrige $\subset (k+1, \ldots, n)$; $\quad \sigma_\gamma = \sigma_\alpha + \sigma_\beta \quad (\gamma = \alpha + \beta)$. Bei festem γ, α ist $\binom{k}{\alpha}\binom{n-k}{\gamma-\alpha}$ die Anzahl dieser σ_γ; die Summe der entsprechenden (σ_γ) ist die W., dass genau γ
Bl. 1v Ereignisse eintreten, von denen genau α zu $1, 2, \ldots, k$ (und $\gamma - \alpha$ zu $k+1, \ldots, n$) gehören. Die Summe aller (σ_γ), die zu Werten α, γ mit $m \leq \alpha \leq \gamma \leq n$ gehören, ist $p_m(1, \ldots, k)$. Analog steht es mit $p_m(\nu_1, \ldots, \nu_k)$. $p_m(\nu_1, \ldots, \nu_k)$ enthält also

$$\sum_{\alpha \geq m} \binom{k}{\alpha}\binom{n-k}{\gamma-\alpha}$$

γ-gliedrige W. der Form (σ_γ); die Summe der rechten Seite in (0) enthält

$$\binom{n}{k} \sum_{\alpha \geq m} \binom{k}{\alpha}\binom{n-k}{\gamma-\alpha}$$

solche (σ_γ), und da sie eine symmetrische Funktion der n Ereignisse ist, enthält sie p_γ in der Vielfachheit

$$\binom{n}{k} \sum_{\alpha \geq m} \binom{k}{\alpha} \binom{n-k}{\gamma-\alpha} : \binom{n}{\gamma}$$

$$= \sum_{\alpha \geq m} \frac{n!}{k!(n-k)!} \; \frac{k!}{\alpha!(k-\alpha)!} \; \frac{(n-k)!}{(\gamma-\alpha)!(n-k-\gamma+\alpha)!} \; \frac{\gamma!(n-\gamma)!}{n!}$$

$$= \sum_{\alpha \geq m} \binom{\gamma}{\alpha} \binom{n-\gamma}{k-\alpha} = c_\gamma$$

(wo c_γ auch von m, k, n abhängt); also

$$\sum p_m(\nu_1, \ldots, \nu_k) = c_m p_m + c_{m+1} p_{m+1} + \cdots + c_n p_n \tag{2}$$

Um (0) zu beweisen, ist

$$c_\gamma \geq \binom{n-m}{k-m} \tag{3}$$

zu zeigen.

Die Binomialkoeffizienten, deren „Nenner“ negativ oder grösser als der „Zähler“ ist, sind 0. Wir haben also in der Formel für c_γ die Summe nach α zu beschränken auf

$$m \leq \alpha \leq \gamma, \quad 0 \leq k - \alpha \leq n - \gamma,$$

also $(m \leq k \leq n, \quad m \leq \gamma \leq n)$: Bl. 2

$$m \leq \alpha \leq \gamma, \quad k + \gamma - n \leq \alpha \leq k.$$

Nun ist $\binom{\gamma}{\alpha} = \frac{\gamma}{\alpha}\binom{\gamma-1}{\alpha-1}$, also Bl. 2v

$$\binom{\gamma}{\alpha} \geq \binom{\gamma-1}{\alpha-1} \geq \binom{\gamma-2}{\alpha-2} \geq \cdots \geq \binom{\gamma-m}{\alpha-m} \quad (m \leq \alpha \leq \gamma),$$

daher

$$c_\gamma \geq \sum_{\alpha \geq m} \binom{\gamma-m}{\alpha-m} \binom{n-\gamma}{k-\alpha}$$

oder, α durch $\alpha + m$ ersetzt,

$$\geq \sum_{\alpha} \binom{\gamma-m}{\alpha} \binom{n-\gamma}{k-m-\alpha} = \sum_{\alpha+\beta=k-m} \binom{\gamma-m}{\alpha} \binom{n-\gamma}{\beta}.$$

Das ist der Koeffizient von x^{k-m} in $(1+x)^{\gamma-m}\,(1+x)^{n-\gamma} = (1+x)^{n-m}$, also $= \binom{n-m}{k-m}$. Daher

$$c_\gamma \geq \binom{n-m}{k-m}.$$

Commentary on Fasz. 778

S. D. Chatterji

with a Postscript by Kai Lai Chung

We restate the result contained in this paper in a slightly different notation. Let $A_1, A_2, \ldots, A_n$ be n measurable sets in some probability space, $\mathbb{P}$ being the underlying probability measure; write X_i for the indicator function of the set A_i (i.e. $X_i(\omega) = 1$ if $\omega \in A_i$, $X_i(\omega) = 0$ if $\omega \notin A_i$) and let, for $m \leq k \leq n$, $1 \leq \nu_1 < \nu_2 < \cdots < \nu_k \leq n$,

$$p_m(\nu_1, \ldots, \nu_k) = \mathbb{P}\left(X_{\nu_1} + \cdots + X_{\nu_k} \geq m\right)$$

Then

$$p_m(1, 2, \ldots, n) \leq \frac{1}{\binom{n-m}{k-m}} \sum p_m(\nu_1, \ldots, \nu_k) \tag{0}$$

the sum being taken over all $\binom{n}{k}$ subsets $\{\nu_1, \ldots, \nu_k\}$ of $\{1, \ldots, n\}$.

The inequality (0) was the main result of Chung's short Comptes Rendus (C. R.) paper of 1940 mentioned by Hausdorff; Hausdorff had seen this inequality (without proof) in the review of Chung's paper in Math. Reviews (exact reference correctly given by Hausdorff) and this led him to construct his own independent proof of (0). Inequality (0) itself is a generalization of a result of Gumbel (1937) (referred to in the titel of Chung's paper) which is the case $m = 1$ in (0); note that the case $m = 1 = k$ of (0) is Boole's subadditivity inequality

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mathbb{P}\left(A_i\right).$$

However, no other case of (0) seems quite so easy, except perhaps the case $k = m$ which can again be proved by the subadditivity of $\mathbb{P}$ and the trivial case $k = n$. Chung's C. R. paper proves (0) completely only for the case $m = 1$, $1 \leq k \leq n$; (i.e. Gumbel's original inequality). Chung then published a more complete paper (*On the probability of the occurrence of at least m events among n arbitrary events*) in the Annals of Math. Statistics 12 (1941), 328–338 with many other results with full proofs. Chung published several papers on allied subjects during the years 1942, 1943, three of them in Annals of Math. Statistics.

Hausdorff's proof of (0) is a little different from Chung's; the basic ingredients in Hausdorff's proof are the probabilities σ_I which, in our notation, are defined by

$$\sigma_I = \mathbb{P}\left\{X_i = 1 \text{ if } i \in I,\ X_i = 0 \text{ if } i \in I^c\right\} = \mathbb{P}\left(\bigcap_{i \in I} A_i \cap \bigcap_{j \in I^c} A_j^c\right)$$

where

$$I = \{\nu_1 < \nu_2 < \cdots < \nu_\gamma\} \subset \{1, 2, \ldots, n\}, \quad I^c = \{1, \ldots, n\} \setminus I$$

(HAUSDORFF denotes σ_I by σ_γ). It is in principle obvious that any probabilistic statement concerning the A_i's must be deducible from the σ_I's; these enter both sides of the inequality (0) and it is a question of careful counting to arrive at the inequality in question.

There are few modern texts which can be cited concerning the subject at hand. FELLER's well-known book [F 1968], chapter IV, gives some of the basic facts but nothing as complicated as (0); FRÉCHET's 1939–1940 monograph [Fr 1940] is the only detailed exposition of this circle of problems; it does not contain the deeper results like the ones in CHUNG's later publications of 1941–1943.

Let us recall that this paper of HAUSDORFF was written less than three months before his death (26 th Jan. 1942) by suicide; we know from various direct testimonies that October–November 1941 was a particularly difficult period for HAUSDORFF when he was fairly isolated with very little access to current scientific literature.

KAI LAI CHUNG (1917–) wrote the papers referred to above in wartime China (during 1939–1943 in Kunming); he later emigrated to the USA where, after obtaining a Ph. d. from Princeton, he established himself as one of the leading probabilists, well-known for his work on sums of independent random variables and Markov Processes. After periods at the Universities of Columbia, Cornell and Syracuse, he settled down as a Professor of Mathematics at Stanford University (California, USA) where he is now professor emeritus. He has kindly agreed to contribute the following post scriptum to this commentary.

Postscript by Kai Lai Chung, August 26, 2003

In Kunming, China, 1940, around the time Paris fell to the Nazis, I received a bunch of reprints from FRÉCHET. It must be his response to my note on an inequality by E. J. GUMBEL that I saw, then wrote a short note extending it, including GUMBEL's theorem, and sent to EMILE BOREL. This Comptes Rendus Note was published in 1940 but I did not see it until I got to Princeton late 1945. HAUSDORFF's posthumous note on my Note was written after he saw only the review of my Note, apparently. He gave a proof of a general statement given in my Note without details. As far as I can see now, it is different from mine. It saddened me that he must have done it “to make TIME pass faster“. He committed suicide shortly after.

I cannot recall precisely now but I must have learned of Hausdorff Space in a course in topology I took in Kunming, at the Southwest Associated Universities (c. 1938–1946). After I came to Princeton I bought HAUSDORFF's book on topology. Chairman SOLOMON LEFSCHETZ told me that its first edition was better than the (third) I had. Later I obtained that first edition, and both books are now on my shelves. I wrote several papers on probabilities of a finite number of events (one of my first two publications). FRÉCHET also has two volumes on the same subject in the Actualités Series. Of course, FRÉCHET was one of the originators of general topology. My old friend Chatterji would know the relationship between the theories of HAUSDORFF and FRÉCHET. I did not meet FRÉCHET until 1960, during a walk in a Japanese garden in Tokyo, 1960. GUMBEL was the first mathematician (statistician) I saw after I arrived in New York. He was also in Tokyo and we had our last supper at the Inn after all the other guests had left.

At Syracuse University I taught once a course using KELLEY's General Topology. I am sure Hausdorff space was a primary topic but I do not remember anymore the numerous generalizations and specializations. TIME must have a stop. It is most kind of Chatterji to let me know the sad "pastime" in the last days of the great mathematician HAUSDORFF.

References

[F 1968] FELLER, W.: *An introduction to probability theory and its applications.* Vol. I (Third ed.). Wiley, New York 1968.

[Fr 1940] FRÉCHET, M.: *Les probabilités associées à un système d'événements compatibles et dépendants* (deux parties). Hermann, Paris 1940, 1943.